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B.M.S COLLEGE FOR WOMEN
BENGALURU-560004

III SEMESTER END EXAMINATION – APRIL 2024

M.Sc. MATHEMATICS -DIFFERENTIAL GEOMETRY
(CBCS Scheme-F+R)

Course Code MM303T | **QP Code: 13003**
Duration: 3 Hours | **Max. Marks: 70**

Instructions: 1) All questions carry equal marks.
2) Answer any five full questions.

1. a) Define directional derivative of a differentiable real-valued function f on E^3. If $v_p = (v_1, v_2, v_3)_p$ is a tangent vector to E^3 then prove that $v_p[f] = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p)$. Using this result compute $v_p[f]$ for $f = e^x \cos y$, $v_p: v = (2, -1, 3), p = (2, 0, 1)$.
   b) If α be a curve in E^3 and f be a differentiable function on E^3, then prove that $\alpha'(t)[f] = \frac{d}{dt}(f \circ \alpha)(t)$.
   c) Let $V = y^2 U_1 - xU_3$ and let $f = xy, g = z^3$. Then compute
   $(i) V[f^2 + g^2]$ $(ii)\ V[fg]$ $(iii)\ V[V[f]]$ $(iv)\ fV[g] - gV[f]$

(5+4+5)

2. a) Evaluate the 1-form $\phi = x^2 dx - y^2 dz$ on the vector fields V, W and $\frac{1}{x}V + \frac{1}{y}W$ where $V = xU_1 + yU_2 + zU_3$ and W= $xy(U_1 - U_3) + yz(U_1 - U_2)$.
   b) Show that every regular curve has a unit speed reparameterization. Also find the unit speed parametrization of the curve $\alpha(t) = (a\cos t, a\sin t, bt), a > 0,\ b \neq 0$.
   c) If $F: E^n \to E^m$ is a mapping and if $\beta = F(\alpha)$ is the image of a curve α in E^n then prove that $\beta' = F_*(\alpha')$.

(4+7+3)

3. a) Explain Frenet apparatus (T, N, B, k, τ) for a unit speed curve $\beta \in E^3$. Express T', N', B' in terms of these.
   b) Define a cylindrical helix. Prove that a regular curve with curvature $k > 0$ is a cylindrical helix if and only if τ / k is constant.
   c) Compute $\nabla_V W$ for $V = (y - x)U_1 + xyU_3,\ W = x^2 U_1 + yzU_3$.

(6+5+3)

4. a) If $A = (a_{ij})$ is the attitude matrix and $W = (w_{ij})$, the matrix of connection forms of a frame field E_1, E_2, E_3 then prove that $W = (dA)A^t$.
b) If S and T are translations, then prove that $ST = TS$ is also a translation.
c) If F is an isometry of E^3such that $F(0) = 0$ then prove that F is an orthogonal transformation. **(5+3+6)**

5. a) Let D be a non-empty open subset of E^3 and let f be a differentiable function defined on D. Then prove that $M = \{(x, y, z) \in E^3 / (x, y) \in D,\ z = f(x, y)\}$ is a simple surface in E^3.
Deduce that the following are simple surfaces:
i. $M: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
ii. $M = \{(x, y, z) \in E^3 / (x^2 + y^2)^2 + 3z^2 = 1\}$.
b) If g is a differentiable real-valued function on E^3 and c is a real constant then prove that $M: g(x, y, z) = c$ of E^3 is a surface provided $dg \neq 0$ at any point of M. Also prove that the gradient vector field $\nabla g = \sum \frac{\partial g}{\partial x_i} U_i$ is a nonvanishing normal vector field on the entire surface M.
(7+7)

6. a) If $\alpha: I \to M$ is a curve in a surface M whose route lies in the image $\mathbf{x}(D)$ of a single patch $\mathbf{x}$ then prove that there exist unique differentiable functions a_1, a_2 on I such that $\alpha(t) = \mathbf{x}(a_1(t), a_2(t))\ \forall\, t \in I$.
b) Obtain the parametrization of cylinder.
c) If $F: M \to N$ be a mapping of surfaces and if ξ and η are forms on N, then prove the following:
i. $F^*(\xi + \eta) = F^*(\xi) + F^*(\eta)$
ii. $F^*(\xi \wedge \eta) = F^*(\xi) \wedge F^*(\eta)$
iii. $F^*(d\xi) = d(F^*\xi)$.
(4+3+7)

7. a) Obtain shape operator of a sphere and prove that every point of sphere is umbilic.
b) If p is an umbilic point of a surface M then show that the shape operator at p is just scalar multiplication by $k = k_1 = k_2$.
b) With usual notations prove
(i) $K = \frac{ln - m^2}{EG - F^2}$ and (ii) $H = \frac{Gl + En - 2Fm}{2(EG - F^2)}$
(3+4+7)

8. a) Show that the image of a Monge patch $\mathbf{x}(u, v) = (u, v, f(u, v))$ is minimal if and only if $(1 + f_u^2) f_{vv} + (1 + f_v^2) f_{uu} - 2 f_u f_v f_{uv} = 0$.
Hence deduce that the image of the patch $\mathbf{x}(u, v) = (u, v, \log\cos v - \log\cos u)$ is minimal with Gaussian curvature $K = \frac{-\sec^2 u \sec^2 v}{(1 + \tan^2 u + \tan^2 v)^2}$.
b) Find all geodesics on the sphere and cylinder. **(8+6)**

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